MA 1118 - Multivariable Calculus
Exam II - Quarter I - AY 02-03

Instructions: Work all problems. Read the problems carefully. Show appropriate work, as partial credit will be given. Closed book. One page (one side) notes permitted. No "blue books" or scientific calculators permitted.

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1. (35 Points) a. Find the equation of the plane that contains the points:

$$(\mathbf{0}, \mathbf{1}, \mathbf{0})\ , \quad (\mathbf{1}, \mathbf{1}, \mathbf{2}) \quad \text{and} \quad (\mathbf{2}, \mathbf{2}, \mathbf{1})$$

Are either of the points $(\mathbf{0}, \mathbf{0}, \mathbf{2})$ or $(-\mathbf{1}, -\mathbf{1}, \mathbf{4})$ on this plane?

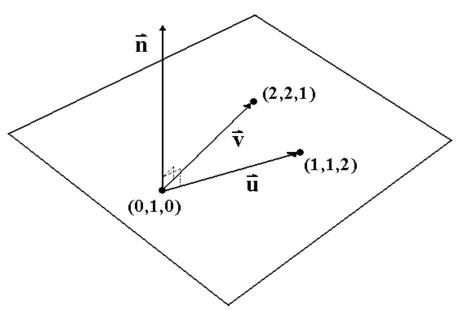
solution:

In order to write the equation for a plane, we need

(1) A point on the plane, and

(2) A vector normal to the plane.

Since we already have three points on the plane, all we need is the normal ($\mathbf{n}$). In this, we can do this simply by finding two non-parallel vectors in the plane, and taking their cross product. The easiest choice for these two vectors are those joining any two pairs of the three given points, e.g.



In this case,

$$\mathbf{u} = (1, 1, 2) - (0, 1, 0) = \mathbf{i} + 2\mathbf{k}$$
$$\mathbf{v} = (2, 2, 1) - (0, 1, 0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

and so

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

solution:

Therefore, the equation of the plane is of the form

$$-2x + 3y + z = d$$

where d can be determined by substituting the value of *any* of the points in on the right, e.g. using $(0, 1, 0)$,

$$d = -2(0) + 3(1) + (0) = 3$$

or the equation of the plane is

$$-2x + 3y + z = 3$$

To determine whether any other given point is on the plane or not, we simply substitute the coordinates of that point in on the left and see if they satisfy the equation, e.g., for $(0, 0, 2)$, we have

$$-2(0) + 3(0) + (2) = 2 \neq 3$$

and so $(0, 0, 2)$ is **not** on the plane. But for $(-1, -1, 4)$,

$$-2(-1) + 3(-1) + (4) = 2 - 3 + 4 = 3$$

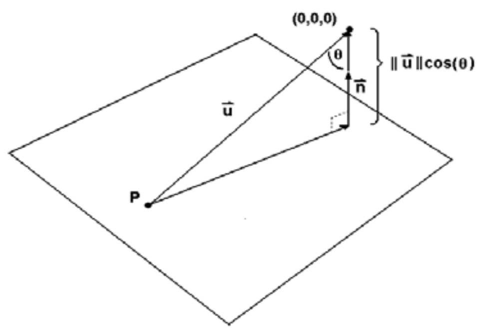
and so $(-1, -1, 4)$ **is** on the plane.

b. Find the distance from the origin to the plane:

$$x + 4y - 3z = 2$$

solution:

We can find this distance if we have the vector ($\mathbf{u}$) from any point in the plane ($\mathbf{P}$) to the origin $(0, 0, 0)$, and the normal vector ($\mathbf{n}$) to the plane, as the following picture describes,



But note that,

$$\| \mathbf{u} \| \cos(\theta) = \frac{\| \mathbf{u} \| \| \mathbf{n} \| \cos(\theta)}{\| \mathbf{n} \|} = \frac{\mathbf{u} \cdot \mathbf{n}}{\| \mathbf{n} \|}$$

In this case, we can find a point on the plane by simply solving for x when y and z are zero, i.e.

$$x + 4(0) - 3(0) = 2 \qquad \Longrightarrow \qquad \mathbf{P} = (2, 0, 0)$$

and so

$$\mathbf{u} = (0, 0, 0) - (2, 0, 0) = -2\mathbf{i}$$

The normal can be found immediately from the equation of the plane, i.e.

$$x + 4y - 3z = 2 \qquad \Longrightarrow \qquad \mathbf{n} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

and so

$$distance = \frac{\mathbf{u} \cdot \mathbf{n}}{\| \mathbf{n} \|} = \frac{(-2, 0, 0) \cdot (1, 4, -3)}{\sqrt{1^2 + 4^2 + (-3)^2}} = \frac{-2}{\sqrt{26}}$$

Since distances are never negative, just change the sign, i.e.

$$distance = \frac{2}{\sqrt{26}} = \frac{\sqrt{26}}{13}$$

2. (30 Points) a. Find an equation for the tangent line to the curve:

$$\mathbf{r}(t) \;=\; \ln(1+t^3)\;\mathbf{i} \;+\; \arctan(t)\;\mathbf{j} \;+\; te^{-3t}\;\mathbf{k} \qquad \text{at} \qquad t=0$$

solution:

To write the equation for a line in space, we need to know a point on the line and the direction of the line. But we also know that the derivative of $\mathbf{r}(t)$, i.e. $\dfrac{d\mathbf{r}}{dt}$ represents the velocity vector for the motion described by $\mathbf{r}(t)$, and therefore lies in the direction of the tangent to the trajectory traced out by $\mathbf{r}(t)$. But

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \frac{3t^2}{1+t^3}\mathbf{i} + \frac{1}{1+t^2}\mathbf{j} + \left(e^{-3t} - 3te^{-3t}\right)\mathbf{k}$$

Therefore, at $t=0$,

$$\begin{aligned}\mathbf{v} = \frac{d\mathbf{r}}{dt} &= \frac{3(0)^2}{1+(0)^3}\mathbf{i} + \frac{1}{1+(0)^2}\mathbf{j} + \left(e^{-3(0)} - 3(0)e^{-3(0)}\right)\mathbf{k} \\ &= (0)\mathbf{i} + (1)\mathbf{j} + (1)\mathbf{k} = \mathbf{j} + \mathbf{k}\end{aligned}$$

Since the tangent vector must pass through the point on the curve at $t=0$, we find the point as

$$\begin{aligned}\mathbf{r}(0) \;&=\; \ln(1+(0)^3)\;\mathbf{i} \;+\; \arctan(0)\;\mathbf{j} \;+\; (0)e^{-3(0)}\;\mathbf{k} \\ &= (0)\mathbf{i} \;+\; (0)\mathbf{j} + (0)\mathbf{k} = \mathbf{0}\end{aligned}$$

and so the equation of the tangent line is

$$\mathbf{0} \;+\; s\,(\mathbf{j}+\mathbf{k}) = s\mathbf{j} + s\mathbf{k}$$

or

$$x = 0 \quad , \quad y = s \quad , \quad z = s$$

b. Find the arc length of the parametric curve:

$$x = \frac{t^3}{3} - t \ , \qquad y = t^2 \ , \qquad 1 \le t \le 3$$

solution:

The arc length a smooth curve with endpoints corresponding to t_0 and t_1 is given by

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

For this curve

$$\frac{dx}{dt} = t^2 - 1 \quad \text{and} \quad \frac{dy}{dt} = 2t$$

Therefore the arc length for $1 \le t \le 3$ is given by

$$\int_1^3 \sqrt{(t^2-1)^2 + (2t)^2}\ dt = \int_1^3 \sqrt{t^4 - 2t^2 + 1 + 4t^2}\ dt$$

$$= \int_1^3 \sqrt{t^4 + 2t^2 + 1}\ dt = \int_1^3 \sqrt{(t^2+1)^2}\ dt$$

$$= \int_1^3 \left(t^2 + 1\right)\ dt = \left(\frac{t^3}{3} + t\right)\Bigg|_1^3 = 12 - \frac{4}{3} = \frac{32}{3}$$

3. (15 Points) Convert the polar equation

$$2\cos^2(\theta) + 5\sin^2(\theta) = 1 + \frac{1}{r^2}$$

to an equation in terms of x and y and identify and sketch the curve represented by this equation.

solution:

Clearing the denominator yields the equivalent equation

$$2r^2\cos^2(\theta) + 5r^2\sin^2(\theta) = r^2 + 1$$

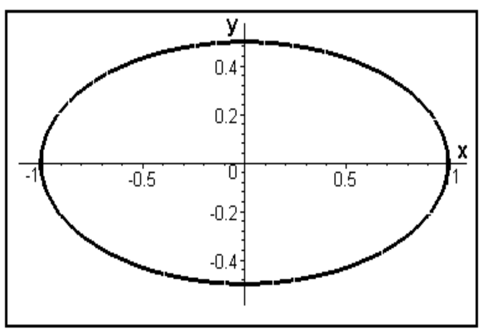
But now, because

$$x = r\cos(\theta) \quad , \quad y = r\sin(\theta) \quad , \quad \text{and} \quad r = \sqrt{x^2 + y^2}$$

this becomes

$$2x^2 + 5y^2 = x^2 + y^2 + 1 \qquad \Longrightarrow \qquad x^2 + 4y^2 = 1$$

But this is just the equation for the ellipse:



4. (20 Points) a. Sketch the surface defined by:

$$x^2 + 2y^2 - z = 4$$

solution:

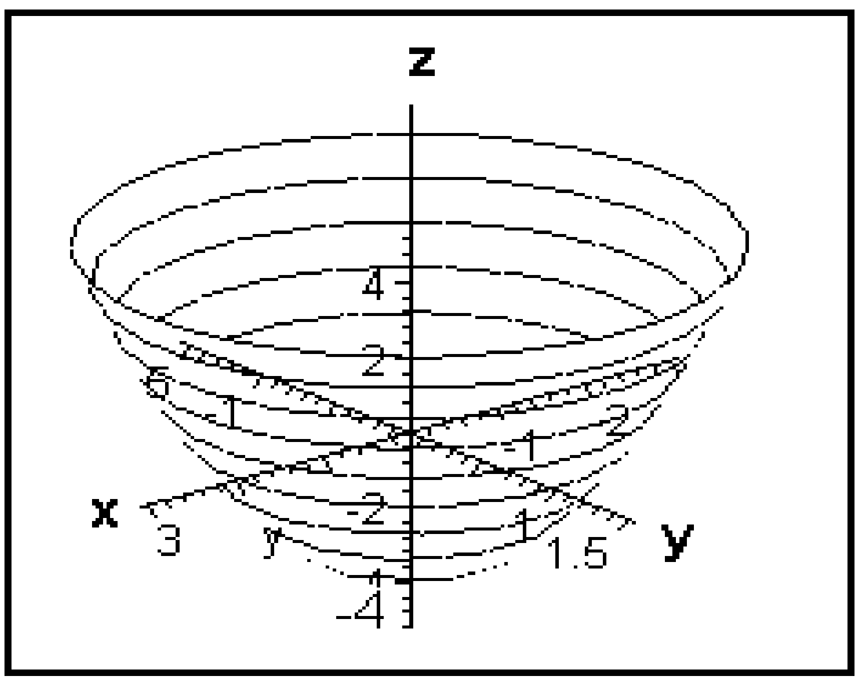
Observe that the equation can be written

$$x^2 + 4y^2 = 4 + z$$

and therefore the cross sections of this surface parallel to the xy plane for $z > -4$ are ellipses. The cross section of the surface in the yz plane is given by

$$4y^2 = 4 + z \qquad \Longrightarrow \qquad z = 4y^2 - 4$$

which is a parabola opening up the z axis, and similarly in the xz plane. Therefore this surface looks like

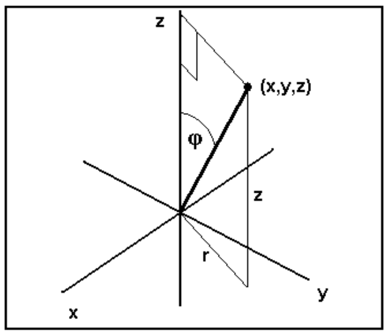


b. Sketch the surface defined in spherical coordinates by:

$$\rho \sin(\phi) = 3$$

solution:

In spherical coordinates, $\rho \sin(\phi)$ represents precisely the distance of the point from the z axis, i.e.



or

$$\rho \sin(\phi) = r = \sqrt{x^2 + y^2}$$

Therefore

$$\rho \sin(\phi) = 3 \qquad \Longrightarrow \qquad x^2 + y^2 = 9$$

i.e. this is a circular cylinder (since z is "missing" from the equation) or radius three about the z axis, i.e.